Financial Economics

2 Contingent Claim Economy

LEC, SJTU

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Overview

Arrow-Debreu
general equilibrium,
welfare theorem,
representative agent

Radner economies
real/nominal assets,
market span,
risk-neutral prob.,
representative good

von Neumann Morgenstern
measures of risk
aversion, HARA class

Finance economy
SDF, CCAPM,
term structure

Data and the Puzzles

Empirical resolutions

Theoretical resolutions

Contingent Claim Economy

- Commodity Space
- Preferences and Utility Function
- Consumer Choices and Maximization
- General Equilibrium
- Social Welfare
- The Representative Agent

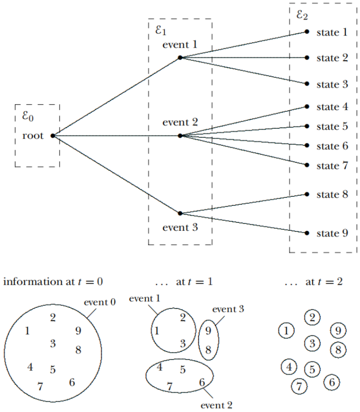
Commodity

- What is a *commodity* (or *good*) ?
  - Physical characteristics
  - Geographical place of availability
  - Time of availability
- Example: An umbrella in London in the summer
- Fourth property: Conditionality
  - A good may or may not be useful conditional on a random (exogenous) event
- Example: An umbrella when it rains v. umbrella when it does not rain –are two different commodities

Events and States

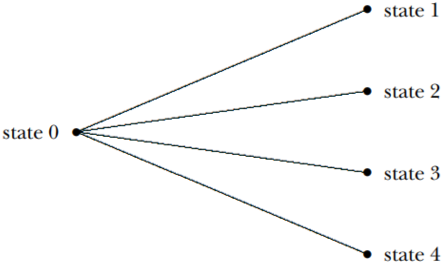
- $\mathcal{S}$ is a finite set with S elements
  - Each element in $\mathcal{S}$ represents a possible *state* of the world
- $\mathcal{E}_t$ is a partition of set $\mathcal{S}$
  - A partition is a collection of non-empty and pairwise disjoint subsets whose union makes up the whole set
  - The elements of $\mathcal{E}_t$ are the *events* that can happen at time t.

Resolution of Uncertainty



- Commodities are event contingent means that for each point in time, the commodity is available if and only if a specific event of this period of time is realized

Two Period Model



- The event tree simplifies considerably in a two-period model
  - Period 1: Complete uncertainty about state of world
  - Period 2: All uncertainty resolved and states of world revealed
- The commodity in two period model is simplified to be state contingent instead of event contingent

Commodities

Definition of a commodity

A complete description of a commodity requires a specification of the following components;

- physical specification,
- place of availability,
- event contingency (or state contingency in a two-period model).

Preferences

- Commodity space
  - let ℓ be the number of different commodities
  - a *consumption bundle* is a point in *commodity space* $\mathbb{R}^\ell$
- Preferences
  - If an agent prefers bundle 1 over bundle 2, we write bundle 1 $\succ$ bundle 2
  - If an agent thinks bundle 1 is <u>at least as good as</u> bundle 2, we write bundle 1 $\succsim$ bundle 2
- Utility function
  - Under some assumptions, preferences can be represented by a utility function, u: $\mathbb{R}^\ell \to \mathbb{R}$, such that

$$\text{bundle 1} \succ \text{bundle 2} \iff u(\text{bundle 1}) > u(\text{bundle 2})$$

Rationality

Definition of rational preference

The preference relation $\succsim$ is rational if it possesses the following two properties:

- Completeness: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$, or both;
- Transitivity: for all $x, y, z \in X$, if $x \succsim y, y \succsim z$, then $x \succsim z$.

Properties of Utility Function

- Continuous
  - no jumps
- Increasing
  - more is better than less
- Strictly quasi-concave (convex preference)
  - some of everything is better than lots of something and nothing of other things
  - indifference curve is convex
- Smooth
  - differentiable arbitrarily many times

Preferences and Ordinal Utility

- Utility function orders the points in commodity space
- Positive transformations of utility functions are equivalent
- Utility function that represents a preference ordering is called ordinal
  - Ordinal utility allows ranking of choices, not levels or differences
  - the utility functions $\sqrt{x_1 x_2}$ and $\ln x_1 + \ln x_2$ are equivalent

More on Rationality

- We have introduced the definition of rational preference. With rational preference (together with other assumption), we can introduce utility function
- What is rational decision?
- In neo-classical framework, rationality means that one chooses the consumption bundle he deems the best among the set of consumption bundles he can afford

Endowment

- Agent's endowment: List of quantities of all commodities you own before any trade takes place
  - Suppose there are ℓ commodities and you own amounts: $\omega_1, \omega_2, \cdots, \omega_\ell$
- Wealth: monetary value of all commodities you own
  - In a perfectly competitive economy, the prices of commodities are $p_1, p_2, \cdots, p_\ell$, then the wealth$= \sum_{c=1}^{\ell} p_c \omega_c$
- Budget constraint: you can consume any combination of goods $x_1, x_2, \cdots, x_\ell$, whose monetary value is not more than your wealth

$$p \cdot x \leq p \cdot \omega, \text{or } p \cdot (x - \omega) \leq 0$$

  - Here $(x - \omega)$ is the excess demand

Maximizing Preference Subject to Budget Constraint

- Revisit *rationality*: choose the bundle one likes best given the constraints imposed. Formally, the problem of an agent becomes

$$\max\{u(x)|p \cdot (x - \omega) \leq 0\}$$

- Additional assumptions on the utility function: (i) strictly convex preference ; (ii) differentiable utility function. (convex and smooth indifference curves)

Kuhn-Tucker Theorem

- Maximization of the agent's consumption problem implies the first order condition, or F.O.C (assuming an interior solution):

$$\partial_c u(x) = \lambda p_c, \text{for } c = 1, \cdots, \ell$$

  where λ is a positive number which is called the Lagrangian multiplier
  λ *measures the marginal utility of wealth*
- Or in vector form:

$$\nabla u(x) = \lambda p$$

  where $\nabla u(x) \coloneqq (\partial_1 u(x), \cdots, \partial_\ell u(x))$ is the gradient of u at x

Geometry of Maximization

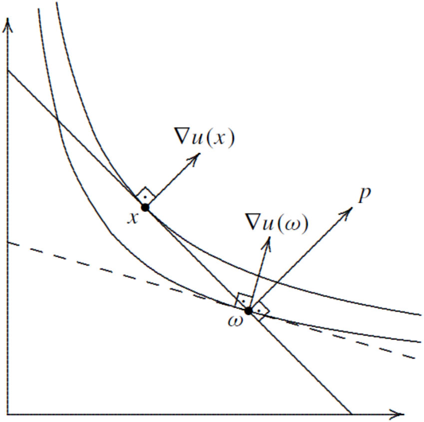


Figure 2.3. *Maximization of a standard preference subject to a budget requires that the gradient of the utility at the maximum be collinear to the price vector, i.e.* $\nabla u(x) = \lambda p$ *for some* $\lambda > 0$.

Geometry of Maximization

- Lagrange multiplier λ measures marginal utility of wealth
- The first order condition means that the gradient of the utility function at the optimal consumption bundle points in the same direction as the price vector
- For any pair of commodities (i, j) we have

$$\frac{\partial_i u(x)}{\partial_j u(x)} = \frac{p_i}{p_j}$$

  Marginal Rate of Substitution (MRS) equals relative price
  - Only the relative prices affect behavior. The price level is irrelevant.

Interest Rates as Relative Prices

- Decision problem of saving for future consumption
  - Suppose your current endowment of wealth$= w$
  - If you save s, you will be able to consume $w - s$ today
  - Let's assume the gross interest rate is ρ, then you will be able to consume ρs tomorrow
  - Your problem becomes:

$$\max_s u(w - s, \rho s)$$

- The first-order condition of this problem is

$$-\partial_1 u + \rho \partial_2 u = 0 \implies \frac{\partial_1 u}{\partial_2 u} = \rho = \frac{p_1}{p_2}$$

  - p_1 is the price of asset that delivers \$1 today; p_2 is the price of asset that delivers \$1 tomorrow
- First Order condition: Real interest rate as MRS between today's and tomorrow's purchasing power

Insurance Premia as Relative Prices

- Suppose you have wealth w
  - In state 1, you are lucky and will keep wealth w
  - In state 2, you are unlucky and will suffer a damage d
- There is an insurance company that offers to cover the loss in exchange for a premium
  - You can choose the coverage rate c (meaning that you get paid cd in state 2) at the cost of $c\mu$ (where μ is the premium of full coverage)
- Your decision problem becomes:

$$\max_{c} u(w - c\mu, w - c\mu - d + cd)$$

- The first-order condition yields

$$\frac{\partial_1 u}{\partial_2 u} = \frac{d - \mu}{\mu}$$

- This can be rearranged into :$\frac{\mu}{d} = \frac{p_2}{p_1 + p_2}$
  where p_1 and p_2 is the price for state-1-contigent and state-2-contigent commodities, respectively

Contingent Claim Economy

- Commodity Space
- Preferences and Utility Function
- Consumer Choices and Maximization
- General Equilibrium
- Social Welfare
- The Representative Agent

General Equilibrium

- GE theory –concerns interaction of optimizing agents through markets
- Questions: Does an equilibrium exist? Is it unique? Are the equilibrium allocations efficient?
- Answers: Yes. Usually not. Yes.
- Finance (or macrofinance - asset pricing for example) is not concerned with existence or equilibrium allocation
- Finance focuses on equilibrium prices and how they relate to utilities (average tastes) and (average) endowments
  - What does "average" tastes mean? Aggregation Problem: to find the representative agent

Abstract Exchange Economy

- Two-period model - M spot assets today and M spot assets in each state of S states tomorrow
- Agent i's utility function: $u_i : \mathbb{R}^{(S+1)M} \to \mathbb{R}$
- Agent i's endowment: $\omega(i) \in \mathbb{R}^{(S+1)M}$
- Collection of all agents in the contingent claim economy is:

$$\{(u_i, \omega(i)) : i = 1, \cdots, I\}$$

- Decision Problem: Agent must choose bundle today $x^0(i)$ and state contingent bundle tomorrow $x^1(i), \ldots, x^S(i)$ such that

$$\max \left\{ u_i(x(i)) \,\middle|\, \sum_{s=0}^{S} p_s \cdot (x^s(i) - \omega^s(i)) \leq 0 \right\}$$

Contingent Claim Economy and Equilibrium

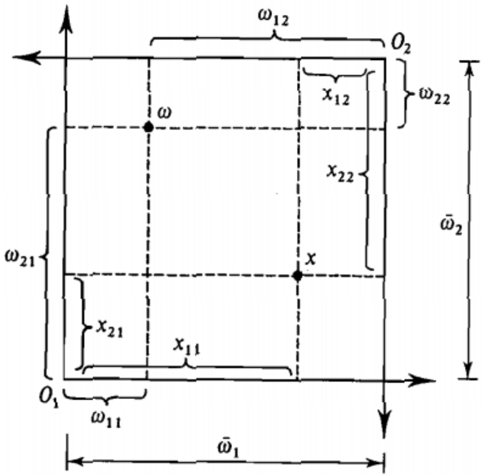
- We have many agents, each one using this optimization rule
- What can happen? Suppose a good is very cheap - many people would like to buy it and few will want to sell it $\Rightarrow$ Demand exceeds supply.
- A competitive equilibrium is a pair (p, x); matrix of prices and collection of consumption bundles; one for each agent, such that for each i, $x(i)$ maximizes i's utility s.t. the budget constraint, given p, and all markets clear (aggregate demand equals aggregate supply for each commodity simultaneously).

$$\sum_{i=1}^{I} x_m^s(i) = \sum_{i=1}^{I} \omega_m^s(i), s = 0, \ldots, S; m = 1, \ldots, M$$

2×2 Exchange Economy

- Consider an economy with two consumers and two goods
  - There are two consumers, $i = 1, 2$, and two goods, $l = 1, 2$
  - The consumption vector of consumer i is $x_i = (x_{1i}, x_{2i})$, with a preference relation $\succsim_i$ over the consumption vector
  - Each consumer is endowed with $\omega_{li} \geq 0$ units of good l
  - The total endowment of good l is $\bar{\omega}_l = \omega_{l1} + \omega_{l2}$, assuming the total endowment of each good is strictly positive
- An allocation in such an economy is $x \in \mathbb{R}^4_+$, which refers to a non-negative consumption vector $x = ((x_{11}, x_{21}), (x_{12}, x_{22}))$
- If for $l = 1, 2$, $x_{l1} + x_{l2} \leq \bar{\omega}_l$ holds, the allocation is feasible; if equality holds, the allocation is non-wasteful

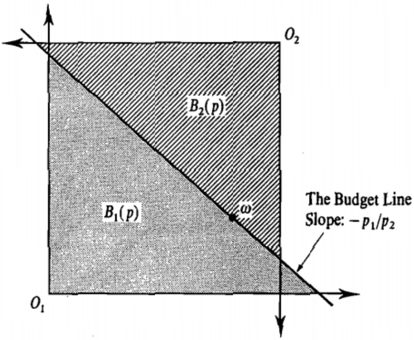
Edgeworth Box



Budget Line in Edgeworth Box

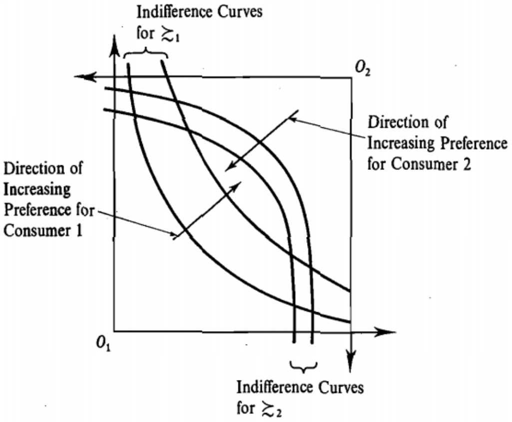
- In general equilibrium theory, wealth is endogenous and depends on the market value of the endowments
- For any price vector $p = (p_1, p_2)$, the budget set of consumer i is

$$B_i(p) = \{x_i \in \mathbb{R}^2_+ : p \cdot x_i \leq p \cdot \omega_i\}$$



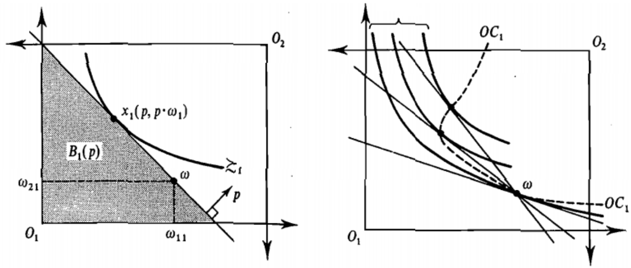
Indifference Curves in Edgeworth Box

- Assume that the preferences $\succsim_i$ of consumers are strictly convex, continuous, and strongly monotonic



Maximize utility

- For a given price vector p, the consumer maximizes his utility subject to a budget constraint and can find the demand function $x_i(p, p \cdot \omega_i)$
- When the price vector p changes, the budget line rotates around the endowment ω, and the curve in which consumer demand varies with price is called the offer curve.
- Offer curves over the endowment point, concentrated in the upper contour of the endowment point, and tangent to the indifference curve at the endowment point

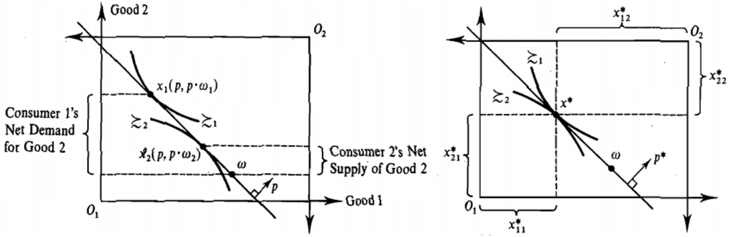


Competitive Equilibrium in the Edgeworth Box

Definition

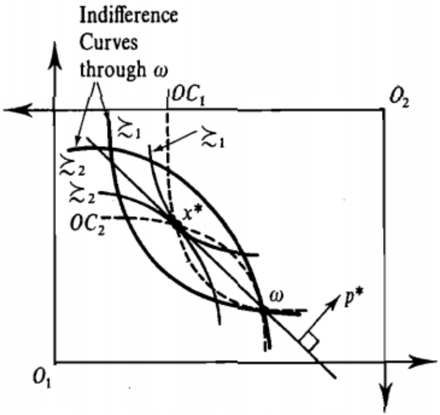
A Walrasian or competitive equilibrium in an Edgeworth box economy is a price vector p^* with a configuration $x^* = (x_1^*, x_2^*)$ in the box such that for$i = 1, 2$

$$x_i^* \succsim_i x_i' \quad \forall x_i' \in B_i(p^*)$$



Determination of competitive equilibrium through offer curves

- Any intersection of the two consumer offer curves other than the endowment point corresponds to an equilibrium configuration



Example of general equilibrium: Cobb-Douglas utility

Assume that each consumer i has a Cobb-Douglas type utility function $u_i = (x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$ with endowment of $\omega_1 = (1,2), \omega_2 = (2,1)$

- The offer curve for consumer 1 is $OC_1(p) = \left(\frac{\alpha(p_1+2p_2)}{p_1}, \frac{(1-\alpha)(p_1+2p_2)}{p_2}\right)$
- The offer curve for consumer 2 is $OC_2(p) = \left(\frac{\alpha(2p_1+p_2)}{p_1}, \frac{(1-\alpha)(2p_1+p_2)}{p_2}\right)$
- At the intersection of the two, there is

$$\frac{\alpha(p_1+2p_2)}{p_1} + \frac{\alpha(2p_1+p_2)}{p_1} = 3$$

from this, we obtain the solution

$$\frac{p_1^*}{p_2^*} = \frac{\alpha}{1-\alpha}$$

Some General Points

- Step #1: How to construct a representative agent
- Step #2: How to go from many commodities into one aggregate commodity namely wealth
- Objective: **one-good and one-agent economy**
    - We want to use this to determine the macroeconomic determinants of asset prices
- Existence of equilibrium is an important issue in GE theory - but not in finance - we will not go into this in this course
    - However here's why this is important: A model should at least guarantee an equilibrium - otherwise it is incomplete
    - GE theory uses things like fixed point theorems to prove these things (see any text like MWG for details)

Pareto Efficiency

- Consider an economy with I agents - aggregate endowment Ω - no markets, prices or budgets
  - People vote how best to distribute endowment - start by randomly assigning an endowment to each agent: $(\omega(1), \ldots, \omega(I))$, s.t. $\sum_{i=1}^{I} \omega(i) = \Omega$
  - Every allocation x that is feasible is proposed: $x = (x(1), \ldots, x(I))$, s.t. $\sum_{i=1}^{I} x(i) \leq \Omega$
  - Voting must be unanimous - as any agent disagrees with the proposed allocation it will not be implemented
- An allocation x is **Pareto efficient** if there is no alternate allocation y that could be unanimously accepted given any initial distribution w
  - Not possible to redistribute consumption among agents so that no one is worse off and at least some one is better off by the redistribution

First Welfare Theorem

- Equilibrium allocations are Pareto efficient. Why?
- Everyone's maximum indifference curve is tangent to the budget hyperplane in equilibrium $\Rightarrow$ no unexploited gains by trade

First Welfare Theorem

Everyone is marginally identical in equilibrium - hence there are no further gains from trade and the equilibrium allocation is Pareto efficient

- In other words, given a competitive equilibrium allocation there is no redistribution that would be accepted unanimously

Social Welfare Function

- Given the utility functions of agents and an aggregate endowment we can generate all Pareto-efficient allocations using a **Social Welfare Function** (SWF)
- SWF is a weighted sum of individual utilities maximized subject to feasibility constraints:

$$U(z) = \max \left\{ \frac{1}{I} \sum_{i=1}^{I} \sigma_i u_i(y(i)) \,\middle|\, \sum_{i=1}^{I} (y(i) - z) \leq 0 \right\}$$

- $\sigma_1, \ldots, \sigma_I, > 0$ are the weights assigned to the respective agents' utility; $z = \Omega/I$ is the mean endowment of each individual
- Setting z equal to the mean endowment of the original economy we can generate every Pareto-efficient allocation by an appropriate choice of weights

Choosing the Competitive SWF-I

- How can we choose $\sigma_1, \ldots, \sigma_I, > 0$, the weights to construct this competitive SWF?
- We use the FOC's of the individual's maximization problem. We know that $\exists \lambda_i > 0$, for each agent s.t.

$$p = \lambda_1^{-1} \nabla u_1(x(1)) = \cdots = \lambda_I^{-1} \nabla u_I(x(I))$$

- λ_i measures the agent's marginal utility of wealth
- The FOC's for the SWF are:

$$\frac{1}{I} \sigma_i \nabla u_i(y(i)) = \mu, i = 1, \ldots, I$$

$$\mu_c \sum_{i=1}^{I} (y_c(i) - z_c) = 0, c = 1, \ldots, (S+1)M$$

Choosing the Competitive SWF-II

- If μ is the vector of Lagrange multipliers, then we search for weights $\sigma_1, \ldots, \sigma_I, > 0$, such that $y = x$ is a solution if $z = \Omega/I$
- Consider the weight $\sigma_i = \lambda_i^{-1}$, substituting x for y and Ω/I for z and bear in mind that the $\mu \gg 0$ (strictly positive)

$$\frac{1}{I}\lambda_i^{-1}\nabla u_i(x(i)) = \mu, i = 1, \ldots, I$$

$$\sum_{i=1}^{I}(x(i) - \Omega/I) = 0$$

- This is a market clearing condition and is satisfied in equilibrium. We know that the equilibrium allocation x satisfies this condition because it is an efficient allocation (by the First Welfare Theorem).
- Thus $\mu = p/I$ is a solution

The Competitive SWF

- We conclude that the equilibrium allocation maximizes a Social Welfare Function that weights agents according to the reciprocal of their marginal utility of wealth.
- Competitive SWF is:

$$U(z) = \max \left\{ \frac{1}{I} \sum_{i=1}^{I} \lambda_i^{-1} u_i(y(i)) \,\middle|\, \sum_{i=1}^{I} (y(i) - z) \leq 0 \right\}$$

- Shadow price is the marginal increase that can be achieved in the objective function if the constraint is eased marginally - the Lagrange multiplier is equal to the shadow price. Enlarging z by dz in the constraint $\sum_{i=1}^{I} y(i) \leq Iz$ eases the constraint by I times dz, hence we get

$$\nabla U(z) = I\mu = p$$

- Equilibrium price $\equiv$ marginal social value of goods.

Some Summing up

- We studied an *abstract contingent claim economy*
- We defined a Pareto efficient allocation and a competitive equilibrium in a contingent claim economy
- A *SWF* is the value of a problem that maximizes a weighted sum of individual utilities subject to the material limitations of the economy
- An allocation is Pareto efficient if and only if it is the solution to some SWF
- A competitive equilibrium is a price-allocation pair in which all markets clear and every agent maximizes utility subject to a budget constraint
- Key result: **First Welfare Theorem - A competitive equilibrium allocation is Pareto efficient**

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Features of Representative Agents

- Economists are interested in aggregate data and equilibrium price, and "society's utility function" that characterize how the market equilibrium is affected by shocks
  - focusing on data that can be observed
  - not necessarily interested in an individual's endowment and decisions
- If we have an economy (u, ω) with I agents - to solve for a competitive equilibrium (p, x) would be cumbersome - we need to do this for every agent
  - If there is only one agent - we know the equilibrium allocation because there is nobody else to trade with - equilibrium prices are just the gradient of this agent's utility function at his endowment point
- Given a multi-agent economy (u, ω) and a competitive equilibrium (p, x) we define a *representative agent* as an artificial agent (u_0, ω_0) such that (p, ω_0) is a competitive equilibrium of this one one-agent economy (u_0, ω_0)
  - The equilibrium allocation is ω_0 or it is a no-trade situation in this one-agent economy

Representative Agents - some discussion

- If we work with a representative agent we lose all information on the *inter-personal equilibrium distribution* - we don't know now who consumes what - however in finance we are not interested in this micro-information
- An arbitrary representative agent: Take any utility function v and a point x where the FOC is achieved: $\nabla v(x) = \lambda p$, then letting $\lambda = 1$, (v, x) is a representative agent
  - Why? Faced with prices p he will not wish to trade, thus forming a one-person general equilibrium
  - However this arbitrary R.A. is not very useful - he has no relation to the original data of the multi-person economy
- As a result of the Pareto efficiency of an equilibrium - everyone is marginally identical in equilibrium $\Rightarrow$ everyone is a representative agent in equilibrium
  - That is to say, for all i, $\nabla u_i(x(i)) = \lambda_i p$; thus, $(u_i, x(i))$ is a representative agent
  - This R.A. is also not very useful - micro data is not available and individual person can make mistakes

Competitive SWF as Representative Agent-I

- We want to construct a representative agent using only the aggregate data of the original economy. How can we do that?
- Using the SWF, the gradient (derivative) of the competitive SWF at the point $z = \Omega/I$ just equals the equilibrium prices:

$$\nabla U \left(\frac{\Omega}{I} \right) = p$$

- Thus $(U, \Omega/I)$ is a representative agent
- This is good news: the R.A.'s endowment is just the per capita endowment in the original economy and we have this data
- What about the construction of U?
- This still requires micro-level data: we need information on the inter-personal distribution of preferences and endowments

Competitive SWF as Representative Agent-II

- What about the construction of U?
- We can estimate U or
- We can assume that everyone has the same utility function u but different endowments $\omega(i)$ and then use data on endowment distributions to compute U
- However we do not need to do this
- Later in this course we will investigate a class of utility functions that can be aggregated without the knowledge of the distribution of endowments

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